

The Algebra of Observables of the closed Bosonic String in $(1 + 3)$ -dimensional Minkowski-Space: updating the structural Analysis

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Abstract

The purpose of the present paper is the communication of some results and observations which shed new light on the algebraic structure of the algebra of string observables both in the classical and in the quantum theory.

A suitable introduction to the construction and investigation of the algebra of observables of the Nambu-Goto string in D -dimensional Minkowski-Space can be found in [1] and [2]. Reference [1] is the starting point for the present communication which will deal only with the case dimension $D = 1 + 3$. Moreover, the mass-square $m^2 = \mathcal{P}^\mu \mathcal{P}_\mu$ will be assumed to be positive and the rest-frame $\mathcal{P}_\mu = m \delta_{\mu,0}$ will be chosen as the reference system. Here, the symbols \mathcal{P}_μ , $\mu \in \{0, 1, 2, 3\}$ denote the components of the energy-momentum vector of the string.

We use the same concepts, methods, definitions and notations as the authors of [1]. In particular, the building units of the present analysis are the monodromy variables $\mathcal{R}_{\mu_1 \dots \mu_M}^t$ with $\mu_i \in \{0, 1, 2, 3\}$, $i \in \{1, \dots, M\}$, $M \in \mathbb{N}$, $M \geq 1$, and we identify \mathcal{R}_μ^t with the components \mathcal{P}_μ of the energy-momentum vector: $\mathcal{R}_\mu^t = \mathcal{P}_\mu$. In the following, the objects $\mathcal{R}_{\mu_1 \dots \mu_M}^t$ are called truncated tensors (*recte*: tensor components) of tensor degree M . The only algebraic relations between them are linear:

$$\mathcal{R}_{\begin{array}{c} \mu_1 \dots \mu_L \\ \mu_{L+1} \dots \mu_M \end{array}}^t = 0, \quad 0 < L < M, \quad M \geq 2,$$

where $\begin{array}{c} \mu_1 \dots \mu_L \\ \mu_{L+1} \dots \mu_M \end{array}$ denotes the sum over the shuffle permutations of the ‘decks’ of indices $\mu_1 \dots \mu_L$ and $\mu_{L+1} \dots \mu_M$, respectively. These relations can be completely resolved by considering the linear span $\mathfrak{R}_{(M)}$ of all truncated tensors of fixed tensor rank M – this linear space has dimension $n(4, M)$ – and by introducing a suitable basis: $\{\mathcal{R}_{\mu_1 \dots \mu_M}^t \mid \text{the sequences } \mu_1 \dots \mu_M \text{ being cyclically minimal when the numerical values for the indices } \mu_i \text{ are inserted}\}$. The symbol $n(4, M)$ stands for

$$n(4, M) = \frac{1}{M} \sum_{D|M} \mu(D) 4^{\frac{M}{D}},$$

where the sum extends over all divisors D of M and where μ denotes the Möbius function. Consider the direct sum $\mathfrak{R} := \bigoplus_{M=1}^{\infty} \mathfrak{R}_{(M)}$. The modified Poisson bracket $\{ , \}_*$,

$$\{\mathcal{R}_{\mu_1 \dots \mu_M}^t, \mathcal{R}_{\nu_1 \dots \nu_N}^t\}_* = \sum_{i=1}^M \sum_{j=1}^N 2 g_{\mu_i \nu_j} (-1)^{M+i+j} \mathcal{R}_{\mu_1 \dots \mu_{i-1} \mu_M \dots \mu_{i+1}}^t \mathcal{R}_{\nu_{j-1} \dots \nu_1 \nu_{j+1} \dots \nu_N}^t \quad \forall M, N \geq 2$$

$$\text{and } \{\mathcal{R}_{\mu}^t, \mathcal{R}_{\nu_1 \dots \nu_N}^t\}_* = 0 \quad \forall N \in \mathbb{N}$$

endows \mathfrak{R} with the structure of a graded Lie algebra:

$$\mathfrak{R} \times \mathfrak{R} \rightarrow \mathfrak{R} : \quad \mathfrak{R}_{(l_1+2)} \times \mathfrak{R}_{(l_2+2)} \rightarrow \mathfrak{R}_{(l_1+l_2+2)}, \quad l_k \in \{-1, 0, 1, \dots\}, k = 1, 2.$$

Before we shall report more detailed properties of the Lie algebra \mathfrak{R} , we turn to one of its Lie subalgebras: the subalgebra of so-called space-like truncated tensors

$$\mathcal{LH} \{ \mathcal{R}_{\mu_1 \dots \mu_M}^t \mid \mu_k \in \{1, 2, 3\} \text{ for } k \in \{1, M\}, \mu_k \in \{0, 1, 2, 3\} \text{ for } k \in \{2, \dots, M-1\}, M \in \mathbb{N} \}.$$

This subalgebra is of paramount importance for the construction and the analysis of the algebra of string observables.

We obtain a basis of this subalgebra in two steps: first, we consider the linear span of the space-like truncated tensors for each tensor rank M separately. We resolve the linear relations between the space-like truncated tensors in each such linear span (i.e. the only algebraic relations among the space-like truncated tensors existing at all), by introducing a suitable basis: for the linear subspace of those truncated tensors of tensor rank M for which *none* of the indices $\{\mu_1, \dots, \mu_M\}$ takes on the value zero, we proceed as before and obtain $n(3, M)$ basis elements. For the linear subspace of the remaining truncated tensors where *at least one* of the indices $\{\mu_2, \dots, \mu_{M-1}\}$ takes on the value zero, we obtain $n(4, M) - n(4, M-1) - n(3, M)$ basis elements by assigning exactly one truncated tensor $\mathcal{R}_{\mu_1 \dots \mu_{j-1} 0 \mu_{j+1} \dots \mu_M}^t$ to every cyclically minimal but not doubly cyclically minimal sequence $0 \nu_2 \dots \nu_M$ such that

$$0 \max (\text{all values obtained from the decks } \mu_{j+1} \dots \mu_M \text{ and } \mu_{j-1} \dots \mu_1 \text{ by a shuffle permutation}) \\ = 0 \nu_2 \dots \nu_M.$$

Here, by definition, a sequence $\mu_1 \dots \mu_M$ is called doubly cyclically minimal if both the sequence $\mu_1 \dots \mu_M$ and the sequence $\mu_2 \dots \mu_M$ are cyclically minimal.

We then consider the direct sum of the former and the latter linear subspace and hence obtain the desired linear span $G_{(M)}$ of dimension $n(4, M) - n(4, M-1)$ for $M > 1$. We define $G_{(1)}$ als the linear span of $\mathcal{R}_{\mu}^t = \mathcal{P}_{\mu}$. Now we form the direct sum $G := \bigoplus_{M=1}^{\infty} G_{(M)}$. The modified Poisson bracket $\{ , \}_*$ endows G with the structure of a graded Lie algebra, embedded in the Lie algebra \mathfrak{R} :

$$G \times G \rightarrow G : \quad G_{(l_1+2)} \times G_{(l_2+2)} \rightarrow G_{(l_1+l_2+2)}.$$

$G_{(l+2)}$ is the germ of the stratum \mathfrak{h}^l of the Poisson algebra (and the quantum algebra) of observables, l corresponding to the degree of the gradation (and the filtration, respectively).

The following facts are known about the algebra G :

- 1.) The direct summands $G_{(l+2)}$, $l = -1, 0, +1, +2, \dots$, are representation spaces of $O(3)$. In fact, the Lie algebra of the infinitesimal generators of $O(3)$ can be identified with $G_{(2)}$.

- 2.) G is *not* finitely generated. It is well known that in every direct summand $G_{(l+2)}$ where l is an odd natural number, there is *at least* one element, for instance: the so-called exceptional element $\sum_{j=1}^3 \mathcal{R}_j \underbrace{0 \dots 0}_l j$, $l = \text{odd natural number}$, which cannot be generated by the (modified) Poisson bracket operation from the elements of the linear spaces $G_{(l'+2)}$ with $l' < l$.

It has been conjectured repeatedly that the set of basis elements of $G_{(l+2)}$, $l \in \{-1, 0, +1\}$ augmented by the set of all exceptional elements of G provides a complete system of generators of the algebra G . Though this conjecture - in a suitably modified form - is correct for the direct sum of the spaces $G_{(l'+2)}$ with $l' \leq 8$, it is wrong for the remainder of G . For the case $l = 9$, the previously mentioned augmented set of generators produces only a subspace of $G_{(9+2)}$ with a dimension falling short of the dimension of $G_{(9+2)}$ by one. One more generator for $G_{(9+2)}$ has to be added to the list, for instance:

$$\frac{1}{2} \sum_{j=1}^3 \sum_{k=1}^3 \mathcal{R}^t \begin{bmatrix} j0 \\ k0 \end{bmatrix} 000 \begin{bmatrix} 0j \\ 0k \end{bmatrix} .$$

Probably, this example marks only the beginning of a new infinite series of generators of the algebra G , possibly the next member of this series being contained in the direct summand $G_{(15+2)}$. Moreover, one must face the unpleasant scenario that the generation of G may require an infinite number of infinite series of generators. However, this structure of the Lie algebra G does not necessarily entail a calamity for the quantum algebra of observables, which is only filtered, in sharp contrast to the classical Poisson algebra of observables, which is graded, and where, indeed, it does create enormous difficulties (see below).

Please note that the generators identified so far beyond those contained in $\bigoplus_{l=-1,0,+1} G_{(l+2)}$ are invariant under the action of the group $O(3)$. This is not merely a coincidence. It can be proved that *all* generators of G beyond those which are contained in $\bigoplus_{l=-1,0,+1} G_{(l+2)}$ may be chosen as $O(3)$ scalars. The proof of this and other related statements made in the present communication will be submitted for publication in an appropriate mathematical-physics journal in due course.

It is not difficult to prove that the Lie algebra G can be decomposed into a semidirect sum of an Abelian algebra \mathcal{A} generated by the exceptional elements $\sum_{j=1}^3 \mathcal{R}_j \underbrace{0 \dots 0}_l j$, $l \in \{1, 3, 5, \dots\}$,

and the Lie algebra G^{rem} generated by the remaining generators of G : $G^{\text{rem}} = \bigoplus_{l=-1}^{\infty} G_{(l+2)}^{\text{rem}}$ (the former algebra acting on the latter one). As for a description of the Lie algebra G^{rem} in terms of generators and generating relations, the situation is as follows: the action of all the exceptional elements on the generators G^{rem} which are contained in the direct summands of $\bigoplus_{l=-1,0,+1} G_{(l+2)}^{\text{rem}}$ is explicitly known and so are all the *independent* generating relations of $\bigoplus_{l=-1}^6 G_{(l+2)}^{\text{rem}}$ (independent if the assistance of the elements of \mathcal{A} is declined) and all the *truly independent* generating relations of $\bigoplus_{l=-1}^6 G_{(l+2)}^{\text{rem}}$ (truly independent if the assistance of the elements of \mathcal{A} is included).

It is conjectured, that, in addition to the relations specifying the actions of the exceptional elements on the infinitely many new generators and the actions of the new generators on each other and on the generators contained in $\bigoplus_{l=-1,0,+1} G_{(l+2)}^{\text{rem}}$, there will arise infinitely many new truly independent generating $J^P = 0^+$ -relations of G^{rem} correlated to the emergence of the new generators, in particular to their respective degrees. The rationale behind this conjecture will be explained below.

Now, returning to the embedding graded Lie algebra $\mathfrak{R} := \bigoplus_{l=-1}^{\infty} \mathfrak{R}_{(l+2)}$, we notice that also each of its direct summands $\mathfrak{R}_{(l+2)}$ carries a linear representation of the group $O(3)$. Thus each

$\mathfrak{R}_{(l+2)}$ can be decomposed into isotypical $O(3)$ components. Making extensive use of this $O(3)$ -covariance and of the information obtained by the analysis of the structure of the embedded Lie algebra G^{rem} up to the year 2003, it has been checked explicitly that $\bigoplus_{l=-1}^5 \mathfrak{R}_{(l+2)}$ is generated by the elements $\mathcal{R}_\mu^t = \mathcal{P}_\mu$, $\mu = 0, 1, 2, 3$ of $\mathfrak{R}_{(-1+2)}$, \mathcal{R}_{0j}^t , $j = 1, 2, 3$ and \mathcal{R}_{jk}^t , $j, k = 1, 2, 3$ of $\mathfrak{R}_{(0+2)}$ and the elements of $\mathfrak{R}_{(1+2)}$ with spin magnetic quantum number zero.

The check - though amounting to a couple of hundred separate computations - did not require much skill. In fact, the vast majority of the separate computations could be carried out with ease in one line.

Also, by the same techniques which were applied before to the Lie algebra G , it can be proved that the only potential candidates for additional generators of \mathfrak{R} must be $O(3)$ -invariant elements.

Judging by this experience, it seems likely to me that the entire Lie algebra \mathfrak{R} is finitely generated by the generators specified above. However, in sharp contrast to the ease of the explicit check mentioned above, a general proof of this speculation may require considerable combinatorical skill. A confirmation of this speculation by a proof would establish a possibly helpful connection between the structural analysis of the algebra of string observables and Hilbert's fourteenth problem [3].

Next, we pass to the symmetric enveloping algebra without unit $S(\mathfrak{R})$, which forms a Poisson algebra. Actually, we are interested in the Poisson subalgebra \mathfrak{h} of $S(\mathfrak{R})$ produced by those polynomials in the truncated tensors at the given fixed point (τ, σ) , $\tau \in \mathbb{R}^1$, $\sigma \in S^1$, which do *not* depend on τ and σ any more. These polynomials are called reparametrization invariants, or for short: invariants. The invariants are of the form

$$\mathcal{Z}_{\mu_1 \dots \mu_M} = \sum_{K=1}^M \mathcal{Z}_{\mu_1 \dots \mu_M}^{(K)} \quad \text{with}$$

$$\mathcal{Z}_{\mu_1 \dots \mu_M}^{(K)} = \frac{1}{K!} \mathfrak{Z}_M \circ \left(\sum_{1 \leq a_1 < \dots < a_{K-1} < M} \mathcal{R}_{\mu_1 \dots \mu_{a_1}}^t \mathcal{R}_{\mu_{a_1+1} \dots \mu_{a_2}}^t \dots \mathcal{R}_{\mu_{a_{K-1}+1} \dots \mu_M}^t \right)$$

where \mathfrak{Z}_M denotes the sum over the cyclic permutations of the indices μ_1, \dots, μ_M . The polynomials $\mathcal{Z}_{\mu_1 \dots \mu_M}^{(K)}$ are also invariants, called homogeneous invariants of tensor rank M and degree of homogeneity K .

The term $\frac{1}{K!} \mathfrak{Z}_M \circ (\mathcal{P}_{\mu_1} \mathcal{P}_{\mu_2} \dots \mathcal{P}_{\mu_{K-1}} \mathcal{R}_{\mu_K \dots \mu_M}^t)$ is called the leading part of $\mathcal{Z}_{\mu_1 \dots \mu_M}^{(K)}$. In the momentum rest frame, the homogeneous invariant $\mathcal{Z}_{\mu_1 \dots \mu_M}^{(K)}$ can be completely reconstructed from just one non-vanishing term of the sum $\frac{1}{K!} \mathfrak{Z}_M \circ (\dots)$ belonging to the leading part: only those terms in the sum $\frac{1}{K!} \mathfrak{Z}_M \circ (\dots)$ survive for which $\mu_{[j]} = \mu_{[j+1]} = \dots = \mu_{[j+k-2]} = 0$ where the symbol $[k]$ denotes the number k modulo M . The sum simplifies to

$$\frac{m^{(K-1)}}{K!} \sum_{j: \mu_{[j]} = \dots = \mu_{[j+k-2]} = 0} \mathcal{R}_{\mu_{[j+k-1]} \dots \mu_{[M+j-1]}}^t, \quad (1)$$

the latter sum producing a non-trivial element of $G_{(l+2)}$ with $l = M - K - 1$ if the sequence $\mu_1 \dots \mu_M$ involves two or more non-zero valued indices, and zero otherwise, unless $M = 1$.

Thus choosing a suitable natural number K for each adequately normalized basis element of $G_{(l+2)}$, one can come up with a homogeneous invariant of tensor rank $l + K + 1$ with degree of homogeneity K and degree of gradation l , called standard invariant [4]. The linear span of the homogeneous invariants of degree l shall be denoted by \mathfrak{h}^l , such that $\mathfrak{h} = \bigoplus_{l=-1}^\infty \mathfrak{h}^l$. Under Poisson bracket operation $\mathfrak{h} \times \mathfrak{h} \rightarrow \mathfrak{h}$, we have

$$\mathfrak{h}^{l_1} \times \mathfrak{h}^{l_2} \rightarrow \mathfrak{h}^{l_1+l_2}.$$

Under tensor multiplication $\mathfrak{h} \times \mathfrak{h} \rightarrow \mathfrak{h}$, we have

$$\mathfrak{h}^{l_1} \times \mathfrak{h}^{l_2} \rightarrow \mathfrak{h}^{l_1+l_2+1}.$$

Thus a reasonably explicit basis (w.r.t. tensor multiplication) of the graded Poisson algebra \mathfrak{h} of algebraically independent elements is available in the form of the complete set of standard invariants.

As for the description of the Poisson algebra \mathfrak{h} in terms of generators and generating relations, the situation corresponds closely to the one of the Lie algebra G : each generator of G brings up a generator of \mathfrak{h} in its train and vice versa, and the same holds for the generating relations. The mapping of a given generator of G with degree l to a generator of \mathfrak{h} is a lift which leaves some freedom, namely the addition of nonlinear degree- l -monomials in the standard invariants.

There is still no general proof that the Poisson algebra \mathfrak{h} possesses the structure corresponding to the decomposition of the Lie algebra G into a semi-direct sum of the abelian Lie algebra \mathcal{A} , generated by the exceptional elements, and the Lie algebra G^{rem} . In fact, in view of the results of [5] on the quantum algebra of string observables, interest has somewhat weakened regarding the existence a Lie algebra \mathfrak{g} with the following properties:

- 1.) \mathfrak{h} coincides with the symmetric algebra obtained from \mathfrak{g} ,
- 2.) \mathfrak{g} is the semi-direct sum of an Abelian algebra \mathfrak{a} generated by suitably chosen counterparts of the generators of \mathcal{A} (in other words, of the exceptional elements), and the Poisson algebra \mathfrak{U} generated by the rest of the generators of \mathfrak{h} : $\mathfrak{g} = \mathfrak{a} \ltimes \mathfrak{U}$, and
- 3.) the generators of \mathfrak{a} act as derivations on the Poisson algebra \mathfrak{U} .

The last comment on the classical algebras \mathfrak{R} and \mathfrak{h} to be made in this paper will concern a rigorous proof of **Proposition 17** in [4]. Roughly, this proposition claims that in the energy-momentum rest frame $\mathcal{P}_\mu = m \delta_{\mu,0}$, $m > 0$, every polynomial \mathcal{P}_{inv} in the truncated tensors $\mathcal{R}_{\mu_1 \dots \mu_M}^t$ which is invariant under reparametrizations, is a polynomial Q (possibly with m -dependent coefficients) in the standard invariants $\mathcal{Z}_{k+1,i}$, $i = 1, \dots, I_{K-1}$, $1 \leq I_{K-1} \leq n(4, K+1) - n(4, K)$, $K \in \mathbb{N}$. Up to now, the ‘Indication of the proof’ in [4] has not been replaced by a complete and conclusive presentation of the argument. The proof announced here follows a different line of argumentation strongly relying on two properties of the standard invariants: their *algebraic* independence [4] and their completeness [6]. Thus it fills a gap and guarantees that the recursive algorithm for the determination of the polynomial Q in the standard invariants for any given polynomial \mathcal{P}_{inv} does not leave any remainder. The said proof also provides an efficient non-recursive algorithm for the determination of Q . The details of this proof can be found in the appendix of the present paper.

Now, let us turn to the construction and structural analysis of a quantum algebra of string observables. In Ref. [7], a recursive deformation routine has been proposed for the passage from the classical algebra of string observables, i.e. the Poisson algebra \mathfrak{h} , to an algebra of quantum string observables, envisaged as the non-commutative enveloping Lie algebra $\hat{\mathfrak{h}}$ generated by the quantum counterparts of the classical generators subject to the quantum counterparts of the classical truly independent generating relations. This deformation routine was organized in terms of successive cycles of step by step increasing degrees $l > 1$. The main ingredients of the routine were the correspondence principle, structural similarity between \mathfrak{h} and $\hat{\mathfrak{h}}$, and consistency (in particular w.r.t. the Jacobi identity). The cycles carried out so far are of degrees 2 to 5 corresponding to

the powers \hbar^4 to \hbar^7 in Planck's constant. For the symbolic computations and their results, see [7] concerning the degrees 2 and 3, [8] and [5] concerning the degrees 4 and 5, respectively.

The above investigations lead to the following revision of the original concepts over the past years:

- (a) the routine is not likely to produce a unique quantum algebra of string observables. Instead, it is likely to produce a *family* of such quantum algebras, labelled by rational valued parameters: first one such parameter f [7], then three such parameters f, g_1 and g_2 , [8], and next possibly even more such parameters.
- (b) the structural similarity requirement applied to the semi-direct splitting $\mathfrak{g} = \mathfrak{a} \ltimes \mathfrak{U}$ cannot be upheld [5].

As to (a): the authors of [8] set out to define the quantum generator $\hat{\mathcal{K}}_0^{(3)}$ – corresponding to the degree-3-generator $\mathcal{B}_0^{(3)}$ of \mathfrak{a} – as a derivation. For that purpose they considered the Poisson actions of $\mathcal{B}_0^{(3)}$ on the generators of \mathfrak{h} contained in \mathfrak{h}^1 . They added to them the most general quantum corrections compatible with the correspondence principle, i.e. sixty-one correction terms of degree l , $0 \leq l \leq 4$ and of appropriate spins and parities. Then on the resulting candidates for the various quantum actions the following consistency requirements were imposed:

- i.) the vanishing of the quantum action of $\hat{\mathcal{K}}_0^{(3)}$ on $\hat{\mathcal{K}}_0^{(1)}$, and
- ii.) the Jacobi identity, which by necessity involves the only free parameter f of the previous cycles of the construction routine of the algebra $\hat{\mathfrak{h}}$.

This reduces the number of free parameters from 61 to 3: apart from f , two new parameters g_1 and g_2 remain undetermined.

The generating relations of \mathfrak{h} carrying a degree l w.r.t. an N-gradation correspond to generating relations of $\hat{\mathfrak{h}}$ with the same degree l , this time, however, w.r.t. an N-filtration and w.r.t. a \mathbb{Z}_2 -gradation ($l = \text{even or odd}$). Realizing this, it is only logical to allow for a similar kind of relationship between the generators of \mathfrak{h} carrying a degree w.r.t. an N-gradation and the corresponding generators of $\hat{\mathfrak{h}}$ carrying the same degree l , this time, however, w.r.t. an N-filtration and w.r.t. a \mathbb{Z}_2 -gradation.

Exploiting the freedom which comes along with this point of view, on the basis of the results of [8] we observe that the additional free parameters g_1 and g_2 – among other free parameters like f allegedly labelling the quantum algebras of string observables – can be completely removed from $\hat{\mathfrak{h}}$, for instance by the following redefinition

$$\hat{\mathcal{K}}_{0[8]}^{(3)} \longrightarrow \hat{\mathcal{K}}_0^{(3)} := \hat{\mathcal{K}}_{0[8]}^{(3)} - g_1 \hat{\mathcal{K}}_{0[7]}^{(1)} - \frac{1}{2} \sqrt{3} g_2 \{ \hat{\mathcal{X}}_1, \hat{\mathcal{X}}_1 \}_0$$

or by an equivalent redefinition (see below). Then for the time being, we are left with only one free parameter, f , labelling the members of the class of quantum algebras of string observables.

As to (b): in [5] the cycle of degree 5 of the quantum deformation routine has been carried out. In particular, the only truly independent classical generating relation of degree 5, a $J^P = 0^+$ relation, has been accordingly deformed. The result is deposited in the electronic archive [math-ph/0210024]. (According to the preceeding comment, without loss of generality the result may be slightly simplified by setting the parameters g_1 and g_2 equal to zero.) The result clearly shows that the quantum algebra of string observables does not even allow for a partition of its

generators into two subsets: the quantum counterparts of the exceptional ones and of the non-exceptional ones, such that the following is true: restricting the filtration degrees to five or less, the monomials in the generators and the (multiple) commutators thereof – (apart from the linear cases) written as (multiple) anticommutators – formed exclusively for the first one of the two subsets are linearly independent of those formed exclusively for the second one. A fortiori, the original hypothesis motivated by an expected structural similarity between \mathfrak{h} and $\hat{\mathfrak{h}}$, must be given up. However, we should not miss the lesson we can draw from the quantum deformation of this $l = 5$, $J^P = 0^+$ classical relation: in striking contrast of the rôle of $\mathcal{B}_0^{(3)}$ in the graded Poisson algebra \mathfrak{h} , where $\mathcal{B}_0^{(3)}$ is a generator, the quantum counterpart $\hat{\mathcal{B}}_0^{(3)}$ is not a generator of $\hat{\mathfrak{h}}$. The quantum relation under discussion implies that $\hat{\mathcal{B}}_0^{(3)}$ can be generated by the generators contained in $\hat{\mathfrak{h}}^0$ and $\hat{\mathfrak{h}}^1$ due to the filtration of $\hat{\mathfrak{h}}$ replacing the gradation of \mathfrak{h} . Actually, $\hat{\mathcal{B}}_0^{(3)}$ may even be defined by this relation, having set g_1 and g_2 equal to zero, since in this relation, $\hat{\mathcal{B}}_0^{(3)}$ appears linearly as a quantum correction. Thus instead of an expected structural similarity we are confronted with a serious dissimilarity (which, however, is not unwelcome at all). If we allow our imagination to roam, then we may even take into consideration a scenario of extreme dissimilarity between \mathfrak{h} and $\hat{\mathfrak{h}}$. In this scenario there exists an infinity of classical $l = \text{odd}$, $J^P = 0^+$ -relations which upon application of the above mentioned quantum deformation routine one by one mutate into formulae defining some or possibly all not yet specified quantum counterparts of the classical exceptional generators $\mathcal{B}_0^{(l-2)}$ (possibly $\mathcal{B}_0^{(l-4)}, \dots$) and of the classical generators looming in the background [cf. the comment at the end of 2. above]. In fact, this may even lead to a finitely generated quantum algebra of string observables $\hat{\mathfrak{h}}$, a thoroughly welcome structure, indeed.

The construction of general (non-trivial) “physical” string states which i) depend exclusively on observable, compatible data, ii) contain the maximal possible information for every ensemble of strings prepared in a pure state and iii) form a separable Hilbert space with a positive-definite scalar product (likewise depending exclusively on the observable, compatible data of the two states involved) carrying a hermitean representation of the Poincaré algebra and of the quantum algebra of string observables is still an open problem. Among other characteristics, a corresponding irreducible hermitean representation would be labelled by a fixed mass and – apart from the trivial vacuum representation – by a fixed (positive) sign of \mathcal{P}^0 as well as by a fixed number $(-1)^F$, $F = 2j$ signalling the exclusive integer-valuedness, respectively, the exclusive half-integer-valuedness of the spins j appearing in its decomposition into isotypical components w.r.t. the Poincaré transformations.

The vacuum state should be invariant under translations and rotations. It is expected to be non-degenerate implying that the representatives of the generators of the algebra of observables reproduce it, or even annihilate it.

There may exist physical string states with non-vanishing energy-momentum and possibly non-vanishing spin which are very similar to those of ordinary elementary particles in so far as they do not have any features characteristic of strings. In other words, there may exist physical states different from the vacuum state, on which representatives of the generators of $\hat{\mathfrak{h}}$ such as the ones contained in $\hat{\mathfrak{h}}^1$ act as annihilators. However, the generating relations of the algebra $\hat{\mathfrak{h}}$ do not allow such states unless their spin is zero or onehalf, the latter option existing only if the (rational valued) free parameter f is set equal to $-8/5$. This is remarkable, as this value for the free parameter f was already suggested before in two seemingly different contexts [9][1].

In this communication, various aspects of the classical and the quantum algebras of string observables \mathfrak{h} and $\hat{\mathfrak{h}}$, respectively, as well as those of the embedding algebra $S(\mathfrak{R})$ of \mathfrak{h} have been discussed. An algorithm which had been applied in the past over and over again has been justified.

A simplification of the generating relations has been noted, thereby reducing the number of free parameters of the quantum algebra $\hat{\mathfrak{h}}$ from 3 to 1. The appearance of an ever increasing number of generators of the classical algebra has been observed as well as the reduction of the number of generators and generating relations of the quantum algebra, contained in $\bigoplus_{l=-1}^6 \hat{\mathfrak{h}}^l$. This observation sheds completely new light on the interrelation between the classical and the quantum algebra, forcing us to revise our concepts for the passage from \mathfrak{h} to $\hat{\mathfrak{h}}$. Briefly, we touched on the open problem of the construction of the hermitean representations of both, the quantum algebra of string observables and the Poincaré algebra, simultaneously and speculated about string states which are very similar to ordinary elementary particle states.

Let me conclude by pointing out two articles, one of which is dealing with the Poincaré covariance of \mathfrak{h} and $\hat{\mathfrak{h}}$ [10], the other one with the massless case for the classical algebra \mathfrak{h} [11]. These articles have not been published, but they have been put on record in the Cornell electronic archive.

Appendix

A Rigorous Proof of Proposition 17 [4]

We consider the case $\text{mass} > 0$.

Proposition 17 [4]: *If division by m is admitted, all invariant charges are polynomials in the standard invariants.*

It is sufficient to prove a more precise formulation of the proposition in the energy momentum rest-frame. In this reference system we consider the basis elements of the Lie algebra \mathfrak{R} , formed by the truncated tensors (*recte*: tensor components) at a fixed base point (τ, σ) , $\tau \in \mathbb{R}^1$, $\sigma \in S^1$: $\mathcal{R}_{\mu_1 \dots \mu_M}^t$ with cyclically minimal sequences $\mu_1 \dots \mu_M$ and $\mathcal{R}_\mu^t = \mathcal{P}_\mu = m \delta_{\mu,0}$. Apart from \mathcal{R}_0^t , these truncated tensors will play the role of the indeterminates in polynomials contained in the enveloping algebra of \mathfrak{R} , the symmetric algebra $S(\mathfrak{R})$. By definition, each such polynomial contains only indeterminates with tensor rank M ranging from 1 up to some $M_{\max} = l_{\max} + 2 < \infty$ and has a finite upper limit for the powers of the monomials involved. Thereby it also has a finite upper limit L_{\max} of the gradation degrees of the monomials involved.

For the following discussion we need the presentation of the Lie algebra \mathfrak{R} in the form of a direct sum of the Lie algebra G and the linear space G' : $G' = \bigoplus_{M=2}^{\infty} G'_{(M)}$, with

$$G'_{(M)} = \mathcal{LH}\{\mathcal{R}_{0\mu_2 \dots \mu_M}^t: \mu_2 \dots \mu_M \text{ cycl. minimal, } \mu_k \in \{0, 1, 2, 3\} \text{ for } k \in \{2, \dots, M\}, M \in \{2, 3, 4, \dots\}\}.$$

Then $\mathfrak{R} = G \oplus G'$.

In addition to the notion of a standard invariant $\mathcal{Z}_{K+1,i}$, the concept of an affiliated “refined standard invariant” $\mathcal{Z}_{K+1,i}^\#$ is useful:

Definition: Let the index $K(K', \dots)$ take on values in the natural numbers and let the index $i(i', \dots)$ label the basis elements $\mathcal{R}_{\mu_1^{(i)} \dots \mu_{K+1}^{(i)}}^t$ ($\mathcal{R}_{\mu_1^{(i')} \dots \mu_{K'+1}^{(i')}}^t, \dots$) contained in $G_{(K+1)}$ ($G_{(K'+1)}, \dots$).

The refined standard invariant $\mathcal{Z}_{K+1,i}^\#$ affiliated with the standard invariant $\mathcal{Z}_{K+1,i}$ is equal to the standard invariant $\mathcal{Z}_{K+1,i}$ plus a polynomial in the standard invariants $\mathcal{Z}_{K'+1,i'}$, where $K' < K$, with possibly m -dependent coefficients such that when all truncated tensors $\mathcal{R}_{\nu_1 \dots \nu_N}^t$ contained in G' are set to zero, only the leading linear term of $\mathcal{Z}_{K+1,i}$ survives.

Clearly, for every standard invariant there does exist a uniquely defined affiliated refined standard invariant, which may in some cases even coincide with the given standard invariant.

Remark: Typically, the dependence of the coefficients is expressed in negative integer powers of m .

Example: $\mathcal{Z}_{2+1,i}^\# := (3-1)! \mathcal{Z}_{001223}^{(3),\#} = (3-1)! \mathcal{Z}_{001223}^{(3)} + \frac{1}{2m} \mathcal{Z}_{012}^{(2)} \mathcal{Z}_{0203}^{(2)}$. The only surviving leading term is $m^2 \mathcal{R}_{1223}^t$, i.e. the leading term of $(3-1)! \mathcal{Z}_{001223}^{(3)}$.

More precise formulation of Proposition 17 [1] in the energy momentum rest-frame:

A given polynomial \mathcal{P}_{inv} (over the field of the complex numbers) in $S(\mathfrak{R})$: a polynomial in the basis elements of \mathfrak{R} as indeterminates, which is invariant under arbitrary orientation-preserving reparametrizations and which does not feature a monomial consisting just of a pure non-negative power of m , is a polynomial (possibly with mass-dependent coefficients) in the refined standard invariants, the latter ones carrying a gradation degree $\leq l_{\max}$.

Proof: Because all basis elements of $\bigoplus_{M=1}^\infty \mathfrak{R}_{(M)}$ are algebraically independent of each other, we are allowed to consider all basis elements not equal to \mathcal{R}_0^t as indeterminates and $\mathcal{R}_0^t = m > 0$ as being a fixed mass. According to the results of section III of [6], the standard invariants $\mathcal{Z}_{K+1,i}$, $i = 1, \dots, I_{K-1}$, $1 \leq I_{K-1} \leq n(4, K+1) - n(4, K)$, $K \in \mathbb{N}$ form a complete system of algebraically independent invariant polynomials contained in $S(\mathfrak{R})$ for fixed mass m , and so do the refined standard invariants $\mathcal{Z}_{K+1,i}^\#$. These refined standard invariants are contained in \mathfrak{h}^{K-1} and, for fixed mass, they are themselves invariant polynomials in the basis elements of $\bigoplus_{M=2}^{K+1} \mathfrak{R}_{(M)}$ with characteristic “surviving” linear terms (consisting each time of one basis elements of the Lie algebra $G_{(K+1)}$ multiplied by the mass m to some positive integer power). The surviving linear terms of the refined standard invariants vary independently of each other and independently of the basis elements of G' . Because all basis elements of $\mathfrak{R}_{(K+1)}$ are algebraically independent of all basis elements of $\mathfrak{R}_{(K'+1)}$ with $K \neq K'$, for fixed mass m , already the refined standard invariants of degree $K-1$, $0 \leq K-1 \leq l_{\max}$, form a complete system of algebraically independent invariant polynomials with the basis elements of $\bigoplus_{M=2}^{l_{\max}+2} \mathfrak{R}_{(M)}$ as indeterminates. Thus \mathcal{P}_{inv} is some function f of the refined standard invariants $\mathcal{Z}_{K+1,i}^\#$ with $1 \leq K \leq l_{\max}+1$, a function still to be determined. Actually, \mathcal{P}_{inv} does not depend on all of the latter refined standard invariants, instead at this stage of the argumentation, a possible dependency on one of the latter refined standard invariants can only exist if its surviving linear term shows up among the indeterminates of \mathcal{P}_{inv} .

It is about time to introduce the following ideal (under tensor multiplication) $J \subset S(\mathfrak{R})$: $J :=$ linear span of all monomials in the basis elements of \mathfrak{R} , which exhibit among their factors at least one basis element, contained in G' .

Actually, we have already called upon this ideal in the definition of $\mathcal{Z}_{K+1,i}^\#$ above.

Subsequently – if desired – all polynomials in the basis elements of \mathfrak{R} which differ by elements of J may be identified. Thereby the refined standard invariants are replaced by their surviving linear terms, the above function f of the refined standard invariants is replaced by the same function f , however this time with the corresponding surviving linear terms as arguments, and \mathcal{P}_{inv} is replaced by a polynomial Q in the basis elements of $\bigoplus_{l'=0}^{l_{\max}} G_{(l'+2)}$, i.e. by a polynomial in the surviving linear terms of refined standard invariants of gradation degree $K - 1$ with $0 \leq K - 1 \leq l_{\max}$.

Again, by the algebraic independence of the surviving linear terms of the refined standard invariants, the function f , still to be determined, with the surviving linear terms of the refined standard invariants (affiliated to the original standard invariants) as arguments instead of the refined standard invariants themselves, must be identical to the polynomial Q with the basis elements of $\bigoplus_{l=1}^{l_{\max}} G_{(l+2)}$ as arguments, i.e. to $\mathcal{P}_{inv} \bmod J$. In particular, the function f must depend on the same arguments as the polynomial Q , it must be a polynomial of the same polynomial degree as Q , and modulo ordering of its arguments, respectively of the arguments of the polynomial Q , it must coincide with Q . Thus the following equation is established:

$$\mathcal{P}_{inv}(\mathcal{R}_{\mu_1 \dots \mu_M}^t : \mu_1, \dots, \mu_M \text{ cyclically minimal}, 2 \leq M \leq l_{\max} + 2) = Q(\mathcal{Z}_{K+1,i}^\#(\mathcal{R}_{\mu_1^{(i)} \dots \mu_{K+1}^{(i)}}^t, \dots)),$$

where $\mathcal{R}_{\mu_1^{(i)} \dots \mu_{K+1}^{(i)}}^t \in G_{K+1}$ is present on the r.h.s. if and only if it already appears in $\mathcal{P}_{inv} \bmod J$. All other \mathcal{R}^t 's on the r.h.s. form part of non-linear monomials displaying at least one factor $\mathcal{R}_{\nu_1^{(i)} \dots \nu_M^{(i)}}^t$ which is contained in $G'_{(M)}$, $2 \leq M \leq K$, $1 \leq K \leq l_{\max} + 1$.

Thereby the Proposition 17 of [4] has been proved.

Remark: As a spin-off, an efficient, non-recursive algorithm for finding the polynomial $Q(\mathcal{Z}_{K+1,i}^\#)$ has been obtained.

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